Temperature Fields Produced by Traveling Distributed Heat Sources

Use of a Gaussian heat distribution in dimensionless form indicates final weld pool shape can be predicted accurately for many welds and materials

BY T. W. EAGAR AND N.-S. TSAI

ABSTRACT. The solution of a travelling distributed heat source on a semi-infinite plate provides information about both the size and the shape of arc weld pools. The results indicate that both welding process variables (current, arc length and travel speed) and material parameters (thermal diffusivity) have significant effects on weld shape. The theoretical predictions are compared with experimental results on carbon steels, stainless steel, titanium and aluminum with good agreement.

Introduction

It has been more than forty years since Rosenthal presented his solution of a travelling point source of heat (Ref. 1) which has been the basis for most subsequent studies of heat flow in welding. Christensen put the results in dimensionless form in order to demonstrate that the solution applies to many materials over wide ranges of heat input (Ref. 2). Since that time, a number of refinements have been offered (Refs. 3-6). Several have attempted to use a more realistic distributed heat source (Refs. 3, 4, 6), but none have solved the entire temperature field for a travelling distributed source. More recently, a general form of the travelling distributed heat source has been offered, but calculations of the thermal field were limited and unexplained (Ref. 7).

Christensen’s experimental results indicate that the Rosenthal solution gives good agreement with the actual weld bead size over several orders of magnitude. However, the scatter can be as much as a factor of three. In addition, the point source solution does not provide any information concerning the shape of the weld pool, since all transverse isotherms are assumed to be semicircular in shape. The question of weld pool shape is of considerable interest at least due to wide variations in welding behavior of heat-to-heat lots (Ref. 8).

One of the purposes of the present study is to determine what shape information can be obtained from the solution of a travelling distributed heat source. Fortunately, several investigators have measured actual heat distributions in arcs on water-cooled copper anodes (Refs. 9, 10). Using these results, it is possible to determine whether the presence of a distributed rather than a point source of heat can explain the range of weld pool shape variation measured by Christensen and others.

In the following sections, a general solution of a travelling distributed heat source is briefly presented. Next, a dimensionless solution of a travelling Gaussian heat distribution is presented with a number of results. These are then compared with experimental weld pool shapes.

It should be emphasized at the outset that this solution retains all but one of the simplifying assumptions used by Rosenthal; the assumptions include absence of convective or radiative heat flow, constant average thermal properties and a quasi-steady state semi-infinite medium. The only change is the use of a distributed rather than a point source of heat. Despite these simplifications, it will be shown that the results not only agree with the Rosenthal solution in the limit, but that they are capable of explaining most of the experimental scatter.

Symbols used when presenting the solutions discussed below are defined in Table 1.

Formulation of the Solution

Rosenthal’s solution for the temperature distribution produced by a steady state point heat source moving on the surface of a semi-infinite plate using the coordinate system shown in Fig. 1 is given by:

\[ T - T_o = \frac{q}{2\pi kR} e^{-\frac{(w+R)^2}{2a}} \]  

(1)

Christensen (Ref. 2) converted this to the dimensionless form:

\[ \theta = n e^{-\frac{(w+R)^2}{R^*}} \]  

(2)

where the operating parameter, \( n \), includes the travel speed, the thermal diffusivity and the net heat input to the workpiece.

The solution to a distributed heat source as shown in Fig. 1 can be formulated by the use of Green’s functions. The steady-state heat conduction equation in a moving coordinate of travel speed \( v \) (Ref. 1) is:

\[ \frac{\partial^2 T^*}{\partial w^2} + \frac{\partial^2 T^*}{\partial y^2} + \frac{\partial^2 T^*}{\partial z^2} - \left( \frac{v}{2a} \right)^2 T^* = -e^{\frac{vw}{2a}} \frac{Q^*}{k} \]  

(3)

where \( T^* = (T - T_o) e^{\frac{vw}{2a}} \) and \( Q^* (w,y,z) \) is the heat source moving at a speed of \( v \).

A derivation of the Green’s function which satisfies equation (3) and suitable
Table 1—List of symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>thermal diffusivity</td>
</tr>
<tr>
<td>c</td>
<td>specific heat</td>
</tr>
<tr>
<td>G</td>
<td>Green's function</td>
</tr>
<tr>
<td>k</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>n</td>
<td>operating parameter (n = qv/4πa^2)</td>
</tr>
<tr>
<td>q</td>
<td>net heat input per unit time (power)</td>
</tr>
<tr>
<td>Q</td>
<td>power distribution</td>
</tr>
<tr>
<td>Q*</td>
<td>heat source moving at a speed of v</td>
</tr>
<tr>
<td>R</td>
<td>distance to the center of arc (R = w^2 + y^2 + z^2)^{1/2}</td>
</tr>
<tr>
<td>R*</td>
<td>dimensionless distance from the center of the arc (R* = [l^2 + p^2 + q^2]^{1/2})</td>
</tr>
<tr>
<td>T</td>
<td>temperature</td>
</tr>
<tr>
<td>T_o</td>
<td>initial temperature</td>
</tr>
<tr>
<td>T_c</td>
<td>critical temperature</td>
</tr>
<tr>
<td>u</td>
<td>dimensionless distribution parameter (u = v/w/2a)</td>
</tr>
<tr>
<td>v</td>
<td>travel speed of arc</td>
</tr>
<tr>
<td>w</td>
<td>distance in x direction in a moving coordinate</td>
</tr>
<tr>
<td>w'</td>
<td>speed</td>
</tr>
<tr>
<td>w</td>
<td>distance in y direction in a moving coordinate</td>
</tr>
<tr>
<td>y</td>
<td>distance in z direction</td>
</tr>
<tr>
<td>σ</td>
<td>distribution parameter</td>
</tr>
<tr>
<td>ρ</td>
<td>density</td>
</tr>
<tr>
<td>δQ</td>
<td>incremental amount of heat</td>
</tr>
<tr>
<td>τ</td>
<td>dimensionless time</td>
</tr>
<tr>
<td>θ</td>
<td>dimensionless temperature</td>
</tr>
<tr>
<td>(T = T_o/T_c)</td>
<td></td>
</tr>
<tr>
<td>ξ</td>
<td>dimensionless distance in the moving coordinate</td>
</tr>
<tr>
<td>ξ</td>
<td>distance in z direction</td>
</tr>
<tr>
<td>ψ</td>
<td>dimensionless distance y</td>
</tr>
<tr>
<td>z</td>
<td>distance in z direction</td>
</tr>
<tr>
<td>oo</td>
<td>infinity</td>
</tr>
</tbody>
</table>

The boundary conditions is given in Appendix A. The resulting Green's function for a distributed surface heat source Q* is:

\[ Q(w,x',y',z') = \frac{2e^{-\frac{v}{2a}(w - w')^2 + (y - y')^2 + (z - z')^2}}{([w - w']^2 + [y - y']^2 + [z - z']^2)^{3/2}} \]  

where (w',x',y',z') is the location of the heat source and (w,y,z) is the point of interest. The temperature distribution is expressed by an integration of the product of the Green's function and the heat distribution over the surface of interest (Ref. 11):

\[ T - T_o = \int Q(w,x',y') \left\{ \frac{e^{-\frac{v}{2a}(w - w')^2 + Q*(w,y')}}{k} \right\} ds' \]  

This is the general solution to a traveling heat source of arbitrary distribution Q*. However, the solution is complicated due to the double integral, and the results are not easily expressed in dimensionless form. Some simplifications can be made if it is assumed that the heat input Q* can be approximated by a Gaussian function,

\[ Q(x,y) = -\frac{q}{2\pi^2} e^{-\left(x^2 + y^2\right)/\sigma^2} \]  

where \( \sigma \) is the amount of heat located at position \((x',y',z')\) at time \( t'\).

The solution of an instantaneous Gaussian heat source is the superposition of a series of point heat source solutions over the distributed region. By substituting the Gaussian distributed heat source for the point heat source \( Q \), this superposition is performed by the integration as shown below and derived in Appendix B.

\[ \Delta \tilde{T}_{f} = \int_{0}^{\infty} dt' \int_{0}^{\infty} dy' \frac{\delta Q}{2\pi^2} \frac{dt'}{\rho c[4\pi a(t - t')]^{3/2}} \]  

\[ \Delta \tilde{T}_{f} = \frac{e^{-\left(x^2 + y^2\right)/\sigma^2}}{2\pi^2} \frac{dt'}{\rho c[4\pi a(t - t')]^{3/2}} \]  

This corresponds to the rise of temperature during a very short time interval from time \( t' \) to \( t' + dt' \) due to an amount of heat \( qdt' \) released on the surface.

When considering the Gaussian heat source travelling with a constant speed \( v \), the total increase of temperature is the sum of all such contributions in the time interval from \( t' \) to \( t' + dt' \). A simpler expression of the solution is expected if the solution is presented in a moving coordinate system with travel speed \( v \). This summation can be carried out again by integration:

\[ T - T_o = \int_{0}^{\infty} dt' \int_{0}^{\infty} dx' \frac{\Delta \tilde{T}_{f}}{\rho c[4\pi a(t - t')]^{3/2}} \]  

\[ \Delta \tilde{T}_{f} = \frac{e^{-\left(x^2 + y^2\right)/\sigma^2}}{2\pi^2} \frac{dt'}{\rho c[4\pi a(t - t')]^{3/2}} \]  

This is similar to the solution obtained by Cline and Anthony (Ref. 13). For simplicity, let \( t'' = t' - t' \), then \( dt'' = dt' \), and \( x - vt' = w + vt'' \), where \( w = x - vt \). Equation (9) can be written as:

\[ T - T_o = \int_{0}^{\infty} dt'' \frac{\Delta \tilde{T}_{f}}{\rho c[4\pi a(t - t'')]^{3/2}} \]  

\[ \Delta \tilde{T}_{f} = \frac{e^{-\left(x^2 + y^2\right)/\sigma^2}}{2\pi^2} \frac{dt''}{\rho c[4\pi a(t - t'')]^{3/2}} \]  

This is the solution of the temperature distribution for a Gaussian distributed heat source moving on a semi-infinite.
plate with no change in phase. The solution may be put in dimensionless form by using the following dimensionless variables: \( \xi \) equals \( \frac{v}{w} \times 2a; \ \psi \) equals \( \frac{v}{w} \times 2a; \ \tau \) equals \( \frac{v^2 \ell}{w} + 2a; \) and \( u \) equals \( \frac{v}{w} \times 2a \)

Equation (10) then reduces to:

\[
\theta = \frac{n}{v^2} \int_0^{\frac{v^2}{2u^2}} \frac{r^{2}}{2} dr + \frac{v^2 + v^2 + v^2 + v^2}{2u^2} = \frac{v^2}{2u^2} \tag{11}
\]

Further solution of this equation requires a numerical procedure.

It is interesting to note that the two primary dimensionless variables that describe the heat source are \( u \) and \( n \). The value \( u \), represents the width of the heat source, while \( n \) is related to the intensity or magnitude of the input energy. Travel speed is an integral part of both \( u \) and \( n \). Typical values of \( u \) and \( n \) vary markedly for different materials and different welding processes as shown schematically in Fig. 2.

Results and Discussion

Results of the Model

In order to represent the solution to equation (11), it is necessary to select values of the distribution parameter, \( \sigma \). Fortunately, Nestor (Ref. 9) and Schoek (Ref. 10) have measured the heat distributions of arcs on water-cooled copper anodes. Although their measured distributions are not true Gaussians, the results can be approximated as Gaussians by a least squares regression of their data to fit a Gaussian distribution, subject to the constraint that the total area under the experimental and the Gaussian curves are equal. This produces a Gaussian distribution of equal total heat input as the experimental distribution. This analysis produces values of \( \sigma \) ranging from 1.6 to 4 mm (0.06 to 0.16 in.); these are in reasonable agreement with the results of Rykaln (Ref. 3).

Qualitatively, it can be argued that \( \sigma \) will increase with increasing current, increasing arc length (i.e., voltage) and increasing tungsten electrode tip angle. The effect of variations in shielding gas composition are less obvious (Ref. 14). A more quantitative discussion of the variation in arc welding conditions will be given in a subsequent publication (Ref. 15).

For the purposes of this paper, values of \( \sigma \) between 1.6 and 4 mm (0.06 to 0.16 in.) will be assumed for gas tungsten arc welding. For travel speeds on the order of 2 mm/s (4.7 ipm), these values of \( \sigma \) will produce values of the dimensionless distribution parameter, \( u \), between 0.4 and 0.8 on carbon steel. In the limit where \( u = 0 \), the Gaussian has no width and the solution to equation (11) reduces to the Rosenthal solution.

It will be noted that \( u \) varies directly with the welding travel speed and inversely with a material condition, i.e., the thermal diffusivity. This means that slow travel speeds and high thermal diffusivity materials such as aluminum or copper alloys are approximated better by the Rosenthal equation than are fast welds on materials such as stainless steel or titanium where \( u \) is larger. For convenience, the results presented here will generally apply to values of \( u = 0 \) (Rosenthal) and 0.4, 0.6 and 0.8.

Figures 3 and 4 show the dimensionless peak temperature distributions in the dimensionless transverse and dimensionless through thickness directions, respectively. Note that the dimensionless temperature is divided by Christensens's operating parameter, \( n \), in order to show all solutions on a single graph. The distribution heat source solution does not predict an infinite centerline temperatures as the Rosenthal solution does; however, it will be noted that the center-line temperature may often exceed the boiling temperature of the metal depending on the values of \( u \) and \( n \). This is an impossible situation. Nonetheless, for values of \( u \) and \( n \) where the evaporative power loss (Ref. 16) is not excessive, the solution given by equation (11) is valid. In other cases, it is still a somewhat better approximation than the Rosenthal solution.

It will be noted that Figs. 3 and 4 provide shape information about the weld pool. At values of \( \theta/n \) corresponding to the melting temperature, Fig. 3 predicts the weld width; this is generally, but not always, greater than that predicted by the Rosenthal solution. Figure 4 predicts the weld depth which is always less than that predicted by the Rosenthal solution.
These points are illustrated further by Figs. 5 and 6 which show the dimensionless weld width and depth as a function of the operating parameter, \( n \). These are lines of constant \( \theta = 1 \). It is easily seen that at low \( n \) (which corresponds to low heat input and travel speed), the weld width is less than the Rosenthal prediction. However, as \( n \) increases, the width rapidly increases to greater than that for the Rosenthal solution, whereas the depth always remains less than the Rosenthal prediction.

Figure 7 is the depth-to-width ratio as a function of operating parameter for different values of the distribution parameter. The wide variations may help to explain at least some of the weld pool depth-to-width ratios which have been reported in the literature when the welding parameters are varied (Ref. 17); however, this model is not capable of predicting the variable penetration at constant process parameters which is caused by convection in the weld pool (Ref. 18). As noted previously, convection is not considered in the solution presented here. Nonetheless, in most cases the model presented here gives a good prediction of the weld pool.

In instances when the convective pattern in the weld pool is constant but process parameters are changed, this model should provide the proper functional relationship between weld process parameters and weld pool shape. When the convective pattern changes from weld to weld, no model which neglects convection such as this one will give an accurate representation of pool shape. In these cases, a much more complicated analysis is required.

Figure 5 also provides an explanation of the changes in weld width with change in arc length as measured by Glickstein (Ref. 14) and confirmed in our laboratory. At low heat inputs the width decreases with increased arc length, while at higher heat inputs the weld width increases with increasing arc length—Fig. 8. This is understood by recognizing that increases in arc length increase the distribution parameter without significantly altering the heat input. Hence, increasing the arc length is essentially an increase in the distribution parameter at constant operating parameter.

As shown in Fig. 5, this results in narrower welds with increasing arc length at low operating parameters but increased weld width with increased arc length at higher operating parameters. At some values of \( n \), where the lines of constant \( u \) intersect, the weld width will be maximum at an intermediate arc length.

A physical explanation for these results is obtained by considering that a very broad heat distribution at low heat inputs will not produce any melting. As the distribution narrows but the net heat input remains the same, the width of the weld increases.

Changes in arc length (i.e., voltage) do not increase the net heat input as longer arcs are less efficient than short ones (Ref. 19). This is due to the fact that most of the heat is transferred by the electrons falling across the anode fall space and condensing in the metal (Ref. 20). Longer plasma columns do not affect the heat transfer processes in the anode fall region. However, they do produce greater radiative heat losses.
input remains constant, the amount of melting increases and the weld width grows. At higher heat inputs there is always sufficient heat for melting, and a narrower distribution will result in a narrower weld more closely approximating the value predicted by the Rosenthal solution.

Figures 9 and 10 show the width and depth of the total transformed zone (fusion zone plus heat-affected zone) as a function of the operating parameter. These graphs correspond to $\vartheta = 0.462$ which is equivalent to $T = 723^\circ C$ ($1333^\circ F$) for carbon steel. Subtracting the data of Figs. 9 and 10 from Figs. 5 and 6 gives the predicted width and depth of the heat-affected zone as shown in Figs. 11 and 12. A distinct maximum in heat-affected zone width occurs at intermediate heat inputs. Such information may be important if one is either trying to minimize the thermal effect in the base metal or if one is trying to use the weld heat-affected zone to alter previous weld passes, as is done with temper beads in heavy section alloy steels. Unfortunately, the maximum in heat-affected zone size is not as pronounced in the depth direction as is seen in Fig. 11.
Figures 13, 14 and 15 give the weld area, total transformed area, and heat-affected zone area, respectively. These areas were obtained by complete solution of equation (11) in two dimensions. The small change in the total transformed area with variations in u is consistent with the fact that, over large distances, the overall heat effect in the metal is proportional to the net heat input—or in the present case—to the operating parameter. When smaller distances, such as the fusion zone, are considered, the deviation may be significant: as seen in Fig. 13. Figures 16 through 18 are provided to show the effect of larger values of u and n. These values apply to materials with low thermal diffusivity such as stainless steel and titanium. Furthermore, estimation of typical values of n for gas tungsten arc welding will show that stainless steel and titanium are often welded in the region where the lines of constant u overlap. This may explain, in part, why more anomalous weld bead shape ratios have been reported for these materials than for carbon steel, aluminum or copper alloys. The shape of the weld bead in stainless steels and titanium is more sensitive to minor variations in the process or in the metal because of inherently large values of u.

Experimental Verification of the Predictions

The accuracy of the model was tested in two ways. In one series of experiments, gas tungsten arc welds were made on carbon steel while varying the welding process variables. In the second test, welds were made on different metals to evaluate the effect of changes in thermal diffusivity.

For the first set of welds, the current, arc length, electrode tip angle, shielding gas composition and travel speed were varied. Changes in the first four of these process variables were correlated with changes in heat distribution on the surface of a water-cooled copper anode by an extensive set of experiments (Ref. 21) similar to the work of Nestor (Ref. 9).

The resulting weld widths, depths and areas are given in Figs. 19, 20 and 21, respectively. The predicted widths and areas as functions of u and n are in good agreement with the theory, although the
depth predictions (Fig. 20) are considerable error. These errors in the depth prediction are due to the assumption of a semi-infinitely thick plate, whereas the true plate thickness was 0.5 in. (12.7 mm). The prediction may be improved by use of the temperature enhancement factor described by Myers (Ref. 22).

Myers used a set of image point heat sources to develop an enhancement factor which estimates the temperature profile of finite thickness plates, based upon the semi-infinite Rosenthal solution. This temperature enhancement factor was used to correct the distributed source theory for the finite thickness of the plate. The results bring the predicted weld pool depth into much better agreement as shown in Fig. 22. The remaining errors in weld pool shape may be attributed either to convection in the weld pool or to depression of the surface due to arc forces, neither of which are considered in the distributed source theory.

For the second test of the distributed source theory, welds were made on stainless steel, titanium and aluminum at a
constant distribution parameter of 

\[ \sigma = 2.4 \text{ mm (0.09 in.)} \]. Due to differences in the thermal diffusivity of these metals, \( \sigma \) is constant, but \( u \) is not constant; thus one would expect a different curve for each material. Figures 23, 24 and 25 show these different curves, based on the weld width. The agreement between theory and experiment is quite good.

It may be questioned why the distributed source theory, which neglects convective and radiative heat transfer as well as temperature dependent properties and phase transformations, can give such good agreement with experiment. The work of Grosh (Ref. 23) showed that the Rosenthal solution is not changed markedly if temperature dependent properties are considered, and Malmuth (Ref. 5) has shown that the latent heat has only a minor effect. Recently, Oprea and Szekely (Ref. 24) showed that convection can, but does not always, play a role in determining weld pool shape. Rykalin (Ref. 25) has shown that the heat lost by radiation is negligible.

Although the distributed source theory presented here is still very imperfect, the solution of this problem clearly shows that the point source assumption is the greatest weakness of the Rosenthal theory. While the distributed source theory cannot always predict the exact weld pool shape, it provides a base line from which to measure the effects of convection, surface depression and possibly evaporate heat loss as they influence weld bead shape. One of the difficulties in utilizing the distributed source theory is a priori prediction of the value of the distribution parameter, \( \sigma \). This important topic is dealt with in a subsequent paper (Ref. 15).

Conclusions

The travelling distributed heat source theory provides the first estimate of weld pool geometry based upon fundamentals of heat transfer. Although a number of simplifying assumptions remain, the agreement between theory and experiment is improved considerably over previous models.

The greatest value of this work does not lie in the ability to predict the absolute size of the weld zone. Rather, the strength of the new theory is that it gives an accurate functional relationship between both process parameters and materials parameters. The theory provides a model that can be used to assess how changes in the process or in the material will influence the weld geometry. Such a model is essential to many
automation and control strategies now being considered for welding processes.

Acknowledgments

The authors wish to express their appreciation to Professor Nils Christensen for several helpful discussions. They are grateful for financial support from the Department of Energy under contract DE-AC02-78ER 94799 A-0003.

References

15. Tsai, N. S., and Eager, T. W. Variation of the heat distribution with welding parameters in gas tungsten arc welding— to be published.

Appendix

A. Derivation of Green's Function

The boundary conditions of the Green's function are the same as those of equation (3)—namely:

1. $\partial T^* / \partial z = 0$ at $z = 0$.
2. $T^* = 0$ at $r = \infty$.

These are the Neumann boundary conditions at the surface $z = 0$, and the Dirichlet boundary condition at infinity. The Green's function must satisfy the equation:

$$\frac{\partial^2 G(r,r')}{\partial r^2} - \left( \frac{1}{2a} \right)^2 \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial G(r,r')}{\partial r} \right) = -\frac{\pi \delta(r-r')}{2a}$$

(A1)

A solution of equation (A1) in spherical coordinates is:

$$G(r-r') = A e^{-\frac{r-r'}{2a}} + B e^{-\frac{r+r'}{2a}}$$

(A2)

The boundary condition that $G(r-r') = 0$ at infinity requires that the first term of equation (A2) vanish. Therefore, the solution can be written as:

$$G(r-r') = \frac{\pi}{2a} e^{-\frac{r-r'}{2a}}$$

(A3)

A symmetric heat source located at the position $(w', y', z')$ is needed in order that the second boundary condition $\partial G(r-r') / \partial z = 0$ at $z = 0$ be satisfied. This imaginary sourceheat leads to the following Green's function:

$$\frac{e^{-\sqrt{(w-w')^2+(y-y')^2+(z-z')^2}}}{\sqrt{(w-w')^2+(y-y')^2+(z-z')^2}}$$

(A4)

If one considers a heat source at the surface $z^* = 0$, the solution reduces to equation (4).
\[
\frac{q d t'}{2 \pi \sigma^2 \rho c (4 \pi a (t - t'))^{3/2}} \exp \left(- \frac{x^2 + y^2}{2 \sigma^2 + 4a(t - t')} \right) \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \exp \left( - \left( \frac{1}{2 \sigma^2} + \frac{1}{4a(t - t')} \right) \right) \left( \left(x' - \frac{2x\sigma^2}{2 \sigma^2 + 4a(t - t')} \right)^2 + \left(y' - \frac{2y\sigma^2}{2 \sigma^2 + 4a(t - t')} \right)^2 \right) \right) \right)
\]

\[
\frac{q d t'}{2 \pi \sigma^2 \rho c (4 \pi a (t - t'))^{3/2} 4a(t - t') + 2\sigma^2} \exp \left(- \frac{x^2 + y^2}{2 \sigma^2 + 4a(t - t')} - \frac{z^2}{4a(t - t')} \right) \right)
\]

\[
\frac{q d t'}{\rho c \pi (4 \pi a (t - t'))^{3/2} 4a(t - t') + 2 \sigma^2} \exp \left(- \frac{x^2 + y^2}{2 \sigma^2 + 4a(t - t')} - \frac{z^2}{4a(t - t')} \right) \right)
\]